

Quantum Channels: Different Representations

Recap: Last week focussed on the Kraus representation

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$$

$$\text{s.t. } \sum_i A_i^\dagger A_i = I$$

This week will cover two other representations

- Stinespring Dilation
- The Choi representation

Stinespring Dilation

Interacting

a system and an environment & then ignoring the environment (tracing it out) induces a ^{non-unitary} channel on the system \rightarrow (we saw this in matrix mechanics problem sheet)

eg. Suppose we start with $\rho_S \otimes |0\rangle\langle 0|_E$

& Evolve under $U = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1 \}$

$$U \rho_S \otimes |0\rangle\langle 0|_E U^\dagger = \begin{aligned} & \rho_S^{00} |0\rangle\langle 0| \otimes U_0 |0\rangle\langle 0| U_0^\dagger \\ & + \rho_S^{11} |1\rangle\langle 1| \otimes U_1 |0\rangle\langle 0| U_1^\dagger \\ & + \rho_S^{01} |0\rangle\langle 1| \otimes U_0 |0\rangle\langle 0| U_1^\dagger \\ & + \rho_S^{10} |1\rangle\langle 0| \otimes U_1 |0\rangle\langle 0| U_0^\dagger \end{aligned}$$

Read as system has a conditional action on the environment

$$\rho_s = \text{Tr}_E (U(\rho_s \otimes |0\rangle\langle 0|_E)U^\dagger)$$

For simplicity let's assume that c is real: $c = c^*$

$$= \rho_s^{00} |0\rangle\langle 0| + \rho_s^{11} |1\rangle\langle 1| + c \rho_s^{01} |0\rangle\langle 1| + c^* \rho_s^{10} |1\rangle\langle 0|$$

We'll now show that this is equivalent to

$$\Rightarrow \equiv p \rho + (1-p) Z \rho Z$$

$$= p \rho +$$

$$(1-p) \rho_s^{00} |0\rangle\langle 0| + \rho_s^{11} |1\rangle\langle 1| - \rho_s^{01} |0\rangle\langle 1| - \rho_s^{10} |1\rangle\langle 0|$$

$$= \rho_s^{00} |0\rangle\langle 0| + \rho_s^{11} |1\rangle\langle 1| + \underbrace{(2p-1)}_c (\rho_s^{01} |0\rangle\langle 1| + \rho_s^{10} |1\rangle\langle 0|)$$

(non-completely)

This is known as the "Dephasing channel"

It kills off coherence turning a quantum state into a classical mixture.

If $c = 0 \Leftrightarrow p = 1/2$ we end up with a perfect classical mixture \rightarrow i.e. the completely dephasing channel we saw last time...

"System acting with measurement / environment such that environment is mapped to orthogonal output states leads to loss of coherence"

\sim Decoherence

But the converse is also true...

- Similarly to how tracing out part of an entangled state lead to a mixed state

So any mixed state can be purified into a part of a larger entangled system

→ Any quantum channel can be written in terms of a unitary acting on an enlarged system.

Check of the larger Hilbert space (again)

Stinespring's Dilation Theorem

Any quantum channel can be written in the form

$$E(\rho) = \text{Tr}_E \left(U (\rho \otimes |0\rangle\langle 0|) U^\dagger \right)$$

\uparrow
 Unitary

Proof: let $E(\rho) = \sum_i A_i \rho A_i^\dagger$ s.t. $\sum_i A_i^\dagger A_i = I$

We need: $U |0\rangle|\psi\rangle = \sum_i |i\rangle A_i |\psi\rangle \quad \forall \psi$

if that's the case:

$$\text{Tr}_E \left(U (|0\rangle\langle 0| \otimes |\psi\rangle\langle \psi|) U^\dagger \right)$$

$$= \sum_{ij} \text{Tr}_E \left(I \otimes A_i (|i\rangle\langle j| \otimes |\psi\rangle\langle \psi|) I \otimes A_j^\dagger \right)$$

$$= \sum_i A_i \rho A_i^\dagger = E(\rho)$$

$$|0\rangle|\psi\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \otimes |\psi\rangle = \begin{pmatrix} |\psi\rangle \\ 0 \\ \vdots \end{pmatrix}$$

$$|i\rangle|\psi\rangle = \begin{pmatrix} 0 \\ |\psi\rangle \\ 0 \\ \vdots \end{pmatrix}$$

$$U |0\rangle | \psi \rangle = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ \vdots \end{pmatrix} \begin{matrix} \text{free to} \\ \text{pick} \\ \text{to} \\ \text{make} \\ \text{unitary} \end{matrix} \begin{pmatrix} | \psi \rangle \\ \vdots \end{pmatrix} = \begin{pmatrix} A_0 | \psi \rangle \\ A_1 | \psi \rangle \\ A_2 | \psi \rangle \\ \vdots \end{pmatrix} = \sum_i |i\rangle A_i | \psi \rangle$$

The Choi Representation

First introduce the notion of vectorisation \rightarrow simple but powerful trick

$$\text{Say } A = \sum_{ij} a_{ij} |i\rangle\langle j|$$

$$| \text{Vec}(A) \rangle = \sum_{ij} a_{ij} |i\rangle |j\rangle$$

$$\text{eg. } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow |00\rangle + |11\rangle$$

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \rightarrow a |01\rangle + b |10\rangle$$

One of the reasons vectorization is useful is because of the following identity:

Central Vectorization Identity

$$\star \quad | \text{Vec}(A \otimes B) \rangle = A \otimes B^T | \text{Vec}(X) \rangle$$

$$\begin{aligned} \text{Proof: } A \otimes B^T | \text{Vec}(X) \rangle &= \sum_{i,j,k,k'} A_{ij} |i\rangle\langle j| \underbrace{(B_{i'j'} |i'\rangle\langle j'|)^T}_{B_{ji}^T} X_{kk'} |kk'\rangle \\ &= \sum_{\substack{i,j,i' \\ k,k'}} A_{ij} B_{ji}^T X_{kk'} |i\rangle |j'\rangle \delta_{ik} \delta_{i'k'} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i,j} \sum_{k,k'} \underbrace{A_{ik} X_{kk'} B_{kj}}_{C_{ij}} |ij\rangle \\
&= \text{Vec}(C) \quad \text{with } C = A X B
\end{aligned}$$

Ok - so why is this useful?

One reason it's useful is we can use it to show that:

Super useful identity:

$$(U^\dagger \otimes I) |\phi^+\rangle = I \otimes U^* |\phi^+\rangle$$

Proof: $|\phi^+\rangle = \frac{1}{\sqrt{2}} \sum_i |ii\rangle = \frac{1}{\sqrt{2}} |\text{Vec}(I)\rangle$

$$(U^\dagger \otimes I) |\phi^+\rangle = \frac{1}{\sqrt{2}} \underbrace{(U^\dagger)}_A \otimes \underbrace{I}_{B^T} |\text{Vec}(I)\rangle$$

$$= \frac{1}{\sqrt{2}} |\text{Vec}(AB)\rangle$$

$$= \frac{1}{\sqrt{2}} |\text{Vec}(U^\dagger)\rangle$$

$$= \frac{1}{\sqrt{2}} |\text{Vec}(AB)\rangle$$

$\begin{matrix} I & \nearrow & U^\dagger \\ & A & \end{matrix}$

$$= (I \otimes U^*) |\phi^+\rangle$$

"Alice apply U to her half a Bell state is equivalent to Bob applying U^* to his...."

This trick can be used to reduce circuit depths.

$$\begin{aligned}
 \text{eg. } & (UV^\dagger \otimes I) |\Phi^+\rangle \\
 &= (U \otimes I)(V^\dagger \otimes I) |\Phi^+\rangle \\
 &= U \otimes V^* |\Phi^+\rangle \quad \leftarrow \begin{array}{l} \text{assuming } U \text{ is depth } 1 \\ \text{ \& } V \text{ is depth } 1 \\ \text{ have reduced depth} \\ \text{ by } 1/2! \end{array}
 \end{aligned}$$

Choi - Jamiołkowski Representation

For any quantum channel \mathcal{E} we define the Choi state associated to the channel as

$$J(\mathcal{E}) := \mathcal{E} \otimes I (\underbrace{|\text{Vec}(I)\rangle\langle\text{Vec}(I)|}_{= |\text{Vec}(I)\rangle\langle\text{Vec}(I)|})$$

you see definitions
both with/without
the factor of dimension
included.

$$\begin{cases} = |\text{Vec}(I)\rangle\langle\text{Vec}(I)| \\ = \sum_i |ii\rangle\langle ii| \\ \propto |\Phi^+\rangle\langle\Phi^+| \end{cases}$$

The Choi state uniquely specifies the quantum channel. If \mathcal{E} has Kraus representation $\{A_i\}$ then the Choi state is given by

$$J(\mathcal{E}) = \sum_i | \text{Vec}(A_i) \rangle \langle \text{Vec}(A_i) |$$

To see this final claim: $J(\mathcal{E}) = \sum_i (A_i \otimes I) |\text{Vec}(I)\rangle\langle\text{Vec}(I)| (A_i^\dagger \otimes I)$

via ~~X~~ $\Rightarrow = \sum_i | \text{Vec}(A_i) \rangle \langle \text{Vec}(A_i) |$

Examples

$\mathcal{E}(\rho)$ = ^{completely} dephasing channel
(on single qubit)

$$A_0 = |0\rangle\langle 0| \\ A_1 = |1\rangle\langle 1|$$

$$|Kee(I)\rangle = |00\rangle + |11\rangle$$

$$\begin{aligned} J(\mathcal{E}) &= (\mathcal{E} \otimes I) (|00\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle\langle 11| + |11\rangle\langle 00|) \\ &= \mathcal{E}(|0\rangle\langle 0|) |0\rangle\langle 0| + \mathcal{E}(|1\rangle\langle 1|) |1\rangle\langle 1| + \mathcal{E}(|0\rangle\langle 1|) |0\rangle\langle 1| + \mathcal{E}(|1\rangle\langle 0|) |1\rangle\langle 0| \\ &= \begin{matrix} \uparrow & & \uparrow \\ |Kee(|0\rangle\langle 0|)\rangle & & |Kee(|1\rangle\langle 1|)\rangle \end{matrix} \end{aligned}$$

i.e. this holds

$$\star = |Kee(A_0)\rangle\langle Kee(A_0)| + |Kee(A_1)\rangle\langle Kee(A_1)|$$

What's the point in the Choi Representation?

- ① often ends up being easier mathematically / computationally to work in terms of States rather than channels.
(e.g. in algorithms for learning / simulating channels)
- ② It provides a way of finding the Kraus operators for a channel

Example $\mathcal{E}(X) = \frac{1}{3} (TX + X^T)$ } what are its Kraus operators?

$$\begin{aligned} J(\mathcal{E}) &= (\mathcal{E} \otimes I) (\sum_{ij} |ii\rangle\langle jj|) \\ &= \sum_{ij} \mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j| \end{aligned}$$

$$= \sum_{i,j} \frac{1}{3} \left(\text{Tr}(\delta_{ij} |i\rangle\langle j|) I + |i\rangle\langle j| I^T \right) \otimes |i\rangle\langle j|$$

$$= \frac{1}{3} \left(\sum_i |i\rangle\langle i| \otimes I + \sum_{i,j} |j\rangle\langle i| \otimes |i\rangle\langle j| \right)$$

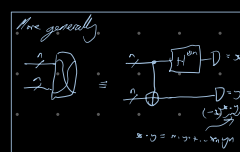
$$= \frac{1}{3} (I + \text{SWAP})$$

Note, if you didn't spot that SWAP could be decomposed like this, you could just find the eigendecomposition of $I + \text{SWAP}$

$$|0\rangle\langle 0| + |0\rangle\langle 0| + |1\rangle\langle 1| + |1\rangle\langle 1| + |2\rangle\langle 2| + |2\rangle\langle 2|$$

$$= \frac{2}{3} (|0\rangle\langle 0| + |0\rangle\langle 0| + |1\rangle\langle 1| + |1\rangle\langle 1| + |2\rangle\langle 2| + |2\rangle\langle 2|)$$

$$= \frac{2}{3} (|00\rangle\langle 00| + |11\rangle\langle 11| + |22\rangle\langle 22|)$$



\Rightarrow 2 different possible choices in Kraus operators

Option 1:

$$|Vee(A_0)\rangle = \sqrt{\frac{2}{3}} |0\rangle = \frac{1}{\sqrt{3}} (|00\rangle + |11\rangle) \Rightarrow A_0 = \frac{1}{\sqrt{3}} I$$

$$|Vee(A_1)\rangle = \sqrt{\frac{2}{3}} |0\rangle \Rightarrow A_1 = \frac{1}{\sqrt{3}} Z$$

$$|Vee(A_2)\rangle = \sqrt{\frac{2}{3}} |1\rangle \Rightarrow A_2 = \frac{1}{\sqrt{3}} X$$

$$\sum_i A_i^\dagger A_i = \frac{2}{3} I + \frac{1}{3} I = I$$

Option 2:

$$|Vee(A_0)\rangle = \sqrt{\frac{2}{3}} |00\rangle$$

$$A_0 = \sqrt{\frac{2}{3}} |0\rangle\langle 0|$$

$$|Vee(A_1)\rangle = \sqrt{\frac{2}{3}} |11\rangle$$

$$A_1 = \sqrt{\frac{2}{3}} |1\rangle\langle 1|$$

$$|Vee(A_2)\rangle = \sqrt{\frac{2}{3}} |1\rangle$$

$$A_2 = \frac{1}{\sqrt{3}} X$$

$$\left. \begin{array}{l} \sum_i A_i^\dagger A_i \\ = \frac{2}{3} I + \frac{1}{3} I \\ = I \end{array} \right\}$$

General Recipe for finding Kraus Operators

- 1) Take \mathcal{E} and compute $\mathcal{J}(\mathcal{E})$
- 2) Find eigendecomposition of $\mathcal{J}(\mathcal{E})$

$$\mathcal{J}(\mathcal{E}) = \sum_k \lambda_k |e_k\rangle\langle e_k|$$

- 3) Find the matrix A_k such that

$$|\text{Vec}(A_k)\rangle = \sqrt{\lambda_k} |e_k\rangle$$

- 4) The matrices $\{A_k\}$ are the smallest set of Kraus operators for \mathcal{E} .

Unitary mixing freedom in Kraus representation

We saw *above* (& previously) that multiple sets of Kraus operators can be used to describe the same channel. This can be understood within the Choi representation.

We have $\mathcal{J}(\mathcal{E}) = \sum_i |\text{Vec}(A_i)\rangle\langle\text{Vec}(A_i)|$

but remember this is just a state & we can write it in its eigendecomposition or just as an

ensemble decomposition.

$$\text{i.e. } \mathcal{J}(\mathcal{E}) = \sum_i |Vec(A_i)\rangle \langle Vec(A_i)|$$

$$\left. \begin{aligned} &= \sum_k \lambda_k |\lambda_k\rangle \langle \lambda_k| \\ &\uparrow \\ &= \sum_i p_i |\phi_i\rangle \langle \phi_i| \end{aligned} \right\}$$

These are related by $\sqrt{p_i} |\phi_i\rangle = \sum_k u_{ik} \sqrt{\lambda_k} |\lambda_k\rangle$
 see lecture 2 ★

elements of a unitary/isometry

★ Recap:

$$|\Psi\rangle_{AB} = \sum_k \sqrt{\lambda_k} |\lambda_k\rangle |u\rangle \leftarrow \begin{array}{l} \text{Purification of} \\ \sum_k \lambda_k |\lambda_k\rangle \langle \lambda_k| \end{array}$$

$$|\Psi'\rangle_{AB} = \sum_i \sqrt{p_i} |\phi_i\rangle |i\rangle \leftarrow \begin{array}{l} \text{Purification of} \\ \sum_i p_i |\phi_i\rangle \langle \phi_i| \end{array}$$

$$|\Psi'\rangle_{AB} = (I \otimes U) |\Psi\rangle_{AB} \leftarrow \begin{array}{l} \text{As } |\Psi\rangle \text{ \& } |\Psi'\rangle \text{ are purifications} \\ \text{of the same state.} \end{array}$$

$$I \otimes \langle i| |\Psi'\rangle_{AB} = \sqrt{p_i} |\phi_i\rangle = \sum_k \sqrt{\lambda_k} \underbrace{\langle i|U|i\rangle}_{u_{ik}} |\lambda_k\rangle$$

$$|Vec(A_i)\rangle = \sqrt{\lambda_i} |\lambda_i\rangle$$

$$\begin{aligned} |Vec(B_i)\rangle &= \sqrt{p_i} |\phi_i\rangle = \sum_k u_{ik} \sqrt{\lambda_k} |\lambda_k\rangle \\ &= \sum_k u_{ik} |Vec(A_k)\rangle \end{aligned}$$

$$\sum_{j'} [B_i]_{jj'} |\phi_j'\rangle = \sum_k u_{ik} [A_k]_{jj'} |\lambda_j'\rangle$$

$$\therefore [B_i]_{jj} = [\sum_k U_{ik} A_k]_{jj}$$

$$= \boxed{B_i = \sum_k U_{ik} A_k}$$

Different possible sets of Kraus operators
are related by unitary/isometric mixing

Eg. for previous example:

$$B_i = \sum_k U_{ik} A_k$$

$$B_0 = \frac{1}{\sqrt{2}} (A_0 + A_1)$$

$$B_1 = \frac{1}{\sqrt{2}} (A_0 - A_1)$$

$$B_2 = A_2$$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$$

clearly
unitary

$$B_0 = \frac{1}{\sqrt{3}} I$$

$$B_1 = \frac{1}{\sqrt{3}} Z$$

$$B_2 = \frac{1}{\sqrt{3}} X$$

$$A_0 = \sqrt{\frac{2}{3}} |0\rangle\langle 0|$$

$$A_1 = \sqrt{\frac{2}{3}} |1\rangle\langle 1|$$

$$A_2 = \frac{1}{\sqrt{3}} X$$