

Quantum Channels: Different Representations

Recap: Last week focussed on the Kraus representation

$$\mathcal{E}(\rho) = \sum_i A_i \rho A_i^\dagger$$

$$\text{s.t. } \sum_i A_i^\dagger A_i = I$$

This week will cover two other representations

- Stinespring Dilation
- The Choi representation

Stinespring Dilation

Interaction

a system and an environment & then ignoring the environment (tracing it out) induces a ^{non-unitary} channel on the system \rightarrow (we saw this in matematica problem sheet)

e.g. Suppose we start with $\rho_s \otimes I_{\mathcal{H}_E}$

& evolve under $U = I_{\mathcal{H}_0} \otimes U_0 + I_{\mathcal{H}_1} \otimes U_1 \quad \{ \}$

$$U \rho_s \otimes I_{\mathcal{H}_E} U^\dagger = \rho_s^{00} I_{\mathcal{H}_0} \otimes U_0 I_{\mathcal{H}_0} U_0^\dagger + \rho_s^{01} I_{\mathcal{H}_0} \otimes U_0 I_{\mathcal{H}_1} U_1^\dagger + \rho_s^{10} I_{\mathcal{H}_1} \otimes U_1 I_{\mathcal{H}_0} U_0^\dagger + \rho_s^{11} I_{\mathcal{H}_1} \otimes U_1 I_{\mathcal{H}_1} U_1^\dagger$$

Read as system has a conditional return on the environment

$$\begin{aligned}
 \rho_s &= T_E (U (\rho_s \otimes 10 \chi_{01E}) U^\dagger) && \text{For simplicity let's assume} \\
 &= \rho_s^{(0)} 10 \chi_{01} + \rho_s^{(1)} 11 \chi_{11} + C \rho_s^{(0)} 10 \chi_{11} + C^* \rho_s^{(1)} 11 \chi_{01} && \text{that } C \text{ is real: } C = C^* \\
 &&& \uparrow \qquad \qquad \qquad \downarrow \\
 &&& := \langle \psi_0 | U_0^\dagger U_0 | 10 \rangle \\
 &&& = \langle \psi_1 | \psi_0 \rangle
 \end{aligned}$$

$$(1-p) \rho_s^{(0)}(0x_0) + \rho_s^{(1)}(1x_1) - \rho_s^{(0)}(0x_1) - * \rho_s^{(1)}(1x_0) \\ = \rho_s^{(0)}(0x_0) + \rho_s^{(1)}(1x_1) + \underbrace{(2p-1)}_C (\rho_s^{(0)}(0x_1) + \rho_s^{(1)}(1x_0)) \\ \text{(non-completely)}$$

This is known as the "Dephasing channel" (non-complex)

It kills off coherence turning a quantum state into a classical mixture.

If $c=0 \Leftrightarrow p = \frac{1}{2}$ we end up with a perfect classical mixture \rightarrow i.e. the completely dephased chord we saw last time ...

"System acting with measurement / environment such that environment is mapped to orthogonal output states leads to loss of coherence."

~ Decoherence

But the converse is also true...

- Similarly to how tracing out part of an entangled state lead to a mixed state

↳ any mixed state can be purified into a part of a larger entangled system

→ Any quantum channel can be written in terms of a unitary acting on an enlarged system.

much of the larger Hilbert space (again)

Stinespring's Dilation Theorem

Any quantum channel can be written in the form

$$E(\rho) = \underset{E}{\text{Tr}} \left(U (\rho \otimes I_{0 \times 0}) U^\dagger \right)$$

↑
Unitary

Proof*: let $E(\rho) = \sum_i A_i \rho A_i^\dagger$ s.t. $\sum_i A_i^\dagger A_i = I$

We need: $U |0\rangle |\psi\rangle = \sum_i |i\rangle A_i |\psi\rangle \quad \forall \psi$

if that's the case:

$$\begin{aligned}
 & \underset{E}{\text{Tr}} (U (|0\rangle\langle 0| \otimes I_{\psi \times \psi}) U^\dagger) \\
 &= \sum_{ij} \underset{E}{\text{Tr}} (I \otimes A_i (|i\rangle\langle j| \otimes I_{\psi \times \psi}) I \otimes A_i^\dagger) \\
 &= \sum_i A_i \rho A_i^\dagger = E(\rho)
 \end{aligned}$$

$$|0\rangle |\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes |\psi\rangle = \begin{pmatrix} |\psi\rangle \\ 0 \end{pmatrix} \quad |i\rangle |\psi\rangle = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \end{pmatrix}$$

$$U |0\rangle |V\rangle = \begin{pmatrix} A_0 & \boxed{\begin{matrix} \text{free to} \\ \text{pick} \\ \text{to} \\ \text{make} \\ \text{unitary} \end{matrix}} \\ A_1 \\ A_2 \\ \vdots \end{pmatrix} \begin{pmatrix} |0\rangle \\ |1\rangle \\ |2\rangle \\ |3\rangle \end{pmatrix} = \begin{pmatrix} A_0 |0\rangle \\ A_1 |1\rangle \\ A_2 |2\rangle \\ \vdots \end{pmatrix} = \sum_i |i\rangle A_i |V\rangle$$

The Choi Representation

First introduce the notion of vectorisation \rightarrow simple but powerful trick

$$\text{Say } A = \sum_{ij} a_{ij} |i\rangle\langle j|$$

$$|\text{Vee}(A)\rangle = \sum_{ij} a_{ij} |ij\rangle$$

$$\text{eg. } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow |00\rangle + |11\rangle$$

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \rightarrow a|01\rangle + b|10\rangle$$

One of the reasons vectorization is useful is because of the following identity:

Central Vectorization Identity

$$\star \quad |\text{Vee}(A \otimes B)\rangle = A \otimes B^T |\text{Vee}(X)\rangle$$

$$\begin{aligned} \text{Proof: } A \otimes B^T |\text{Vee}(X)\rangle &= \sum_{ijkl} A_{ij} |i\rangle\langle j| \underbrace{(B_{kl} |l\rangle\langle k|)^T}_{B_{kl}^T |l\rangle\langle k|} \chi_{ul} |kk\rangle \\ &= \sum_{ijkl} A_{ij} B_{kl}^T \chi_{ul} |ij\rangle \delta_{ul} \delta_{ik} \end{aligned}$$

$$\begin{aligned}
 &= \underbrace{\sum_{i,j} \sum_{k,k'} A_{ik} X_{kk'} B_{kj}}_{C_{ij}} |i,j\rangle \\
 &= \text{Vee}(C) \quad \text{with } C = A X B
 \end{aligned}$$

Ok - so why is this useful?

One reason it's useful is we can use it to show that:

Super useful identity:

$$(U^\dagger \otimes I) |\phi^+\rangle = I \otimes V^* |\phi^+\rangle$$

Proof: $|\phi^+\rangle = \frac{1}{\sqrt{2}} \sum_i |ii\rangle = \frac{1}{\sqrt{2}} |\text{Vee}(I)\rangle$

$$(U^\dagger \otimes I) |\phi^+\rangle = \frac{1}{\sqrt{2}} (U^\dagger \otimes I) |\text{Vee}(I)\rangle$$

\uparrow \uparrow
 A B^\dagger

$$= \frac{1}{\sqrt{2}} |\text{Vee}(AB)\rangle$$

$$= \frac{1}{\sqrt{2}} |\text{Vee}(U^\dagger)\rangle$$

$$= \frac{1}{\sqrt{2}} |\text{Vee}(A B)|$$

\uparrow \uparrow
 I U^\dagger

$$= (I \otimes V^*) |\phi^+\rangle$$

"Alice applying V to her half a Bell state is equivalent to Bob applying V^* to his..."

This trick can be used to reduce circuit depth.

$$\begin{aligned}
 \text{eg. } & (U V^* \otimes I) | \phi^+ \rangle \\
 &= (U \otimes I) (V^* \otimes I) | \phi^+ \rangle \\
 &= U \otimes V^* | \phi^+ \rangle \quad \left(\begin{array}{l} \text{assuming } U \text{ is depth } 1 \\ \text{& } V \text{ is depth } 1 \end{array} \right) \\
 & \quad \text{have reduced depth} \\
 & \quad \hookrightarrow \frac{1}{2}!
 \end{aligned}$$

Choi - Jamiołkowski Representation

for any quantum channel \mathcal{E} we define the Choi state associated to the channel as

$$\mathcal{J}(\mathcal{E}) := \mathcal{E} \otimes I \left(\underbrace{| \text{Vee}(I) \rangle \langle \text{Vee}(I) |}_{{}^{\text{you see definitions}} \atop {}^{\text{both with/without}} \atop {}^{\text{the factor of dimension included}}} \right)$$

$$\begin{aligned}
 & \text{you see definitions} \\
 & \text{both with/without} \\
 & \quad \left\{ \begin{array}{l} = | \text{Vee}(I) \rangle \\ = \sum_i | i \rangle \langle i | \\ \propto | \phi^+ \rangle \end{array} \right.
 \end{aligned}$$

The Choi state uniquely specifies the quantum channel. If \mathcal{E} has Kraus representation $\{A_i\}$ then the Choi state is given by

$$\mathcal{J}(\mathcal{E}) = \sum_i | \text{Vee}(A_i) \rangle \langle \text{Vee}(A_i) |$$

To see this final claim: $\mathcal{J}(\mathcal{E}) = \sum_i (A_i \otimes I) | \text{Vee}(I) \rangle \langle \text{Vee}(I) | (A_i^* \otimes I)$

via $\star \quad \hookrightarrow \sum_i | \text{Vee}(A_i) \rangle \langle \text{Vee}(A_i) |$

Examples

$\mathcal{E}(\rho) = \text{V dephasing channel}$
(on single qubit)

$$A_0 = 10 \times 01 \\ A_1 = 12 \times 12$$

$$|\text{Vee}(I)\rangle = |100\rangle + |111\rangle$$

$$\begin{aligned} \mathcal{J}(\mathcal{E}) &= (\mathcal{E} \otimes \mathcal{I}) (|100\rangle\langle 001| + |111\rangle\langle 111| + |100\rangle\langle 111| + |111\rangle\langle 001|) \\ &= \mathcal{E}(10 \times 01) |10\rangle\langle 01| + \mathcal{E}(12 \times 12) |11\rangle\langle 12| + \mathcal{E}(10 \times 12) |10\rangle\langle 12| + \mathcal{E}(12 \times 01) |11\rangle\langle 01| \\ &= |100\rangle\langle 001| + |111\rangle\langle 111| \quad \equiv \\ &\quad \uparrow \quad \uparrow \\ &\quad |\text{Vee}(10 \times 01)| \quad |\text{Vee}(12 \times 12)| \\ \star &= |\text{Vee}(A_0)\rangle \langle \text{Vee}(A_0)| + |\text{Vee}(A_1)\rangle \langle \text{Vee}(A_1)| \end{aligned}$$

What's the point in the Choi Representation?

- ① Often ends up being easier mathematically / computationally to work in terms of states rather than channels.
(e.g. in algorithms for learning / simulating channels)
- ② It provides a way of finding the Kraus operators for a channel

Example $\mathcal{E}(X) = \frac{1}{3} (\text{Tr}(X) \mathcal{I} + X^T) \quad \left\{ \begin{array}{l} \text{what are its} \\ \text{Kraus operators?} \end{array} \right.$

$$\begin{aligned} \mathcal{J}(\mathcal{E}) &= (\mathcal{E} \otimes \mathcal{I}) \left(\sum_{ij} |ii\rangle\langle jj| \right) \\ &= \sum_{ij} \mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j| \end{aligned}$$

$$\begin{aligned}
 &= \sum_{ij} \frac{1}{3} \left(\text{Tr}(\cancel{X_{ij}}) I + \langle i j | \cancel{X_{ij}}^T \right) \otimes \cancel{X_{ij}} \\
 &= \frac{1}{3} \left(\sum_i \langle i i | \otimes I + \sum_{ij} \langle i j | \cancel{X_{ij}} \right) \\
 &= \frac{1}{3} (I + \text{SWAP})
 \end{aligned}$$

Note, if you didn't spot that SWAP could be decomposed like this, you could just find the eigen-decomp. of $I + \text{SWAP}$

$$\begin{aligned}
 &= \frac{2}{3} (| \phi^+ \rangle \langle \phi^+ | + | \phi^- \rangle \langle \phi^- | + | \psi^+ \rangle \langle \psi^+ | - | \psi^- \rangle \langle \psi^- |) \\
 &= \frac{2}{3} (| 00 \rangle \langle 00 | + | 11 \rangle \langle 11 | + | \psi^+ \rangle \langle \psi^+ |)
 \end{aligned}$$

More generally,

$$\frac{1}{2} (| \psi \rangle \langle \psi | - I) = \frac{1}{2} \left(\frac{1}{\sqrt{m}} \sum_{i=1}^m | i \rangle \langle i | - I \right)$$

* $i = 0, 1, \dots, m-1$

\Rightarrow 2 different possible choices in Kraus operators

Option 1:

$$| \text{Vee}(A_0) \rangle = \sqrt{\frac{2}{3}} | \phi^+ \rangle = \sqrt{\frac{1}{3}} (| 00 \rangle + | 11 \rangle) \Rightarrow A_0 = \sqrt{\frac{1}{3}} I$$

$$| \text{Vee}(A_1) \rangle = \sqrt{\frac{2}{3}} | \phi^- \rangle \Rightarrow A_1 = \sqrt{\frac{1}{3}} Z$$

$$| \text{Vee}(A_2) \rangle = \sqrt{\frac{2}{3}} | \psi^+ \rangle \Rightarrow A_2 = \sqrt{\frac{1}{3}} X$$

$\text{Note } \sum_i A_i^* A_i = I, \sqrt{\frac{1}{3}} I = I$ ✓

Option 2:

$$| \text{Vee}(A_0) \rangle = \sqrt{\frac{2}{3}} | 00 \rangle \quad A_0 = \sqrt{\frac{2}{3}} | 00 \rangle \langle 00 |$$

$$| \text{Vee}(A_1) \rangle = \sqrt{\frac{2}{3}} | 11 \rangle \quad A_1 = \sqrt{\frac{2}{3}} | 11 \rangle \langle 11 |$$

$$| \text{Vee}(A_2) \rangle = \sqrt{\frac{2}{3}} | \psi^+ \rangle \quad A_2 = \sqrt{\frac{1}{3}} X$$

$$\begin{aligned}
 \sum_i A_i^* A_i &= \frac{2}{3} I + \frac{1}{3} I \\
 &= I
 \end{aligned}$$

General Recipe for finding Kraus Operators

- 1) Take \mathcal{E} and compute $\mathcal{J}(\mathcal{E})$
- 2) Find eigendecomposition of $\mathcal{J}(\mathcal{E})$

$$\mathcal{J}(\mathcal{E}) = \sum_k \lambda_k |e_k\rangle\langle e_k|$$

- 3) Find the matrix A_k such that

$$| \text{Vee}(A_k) \rangle = \sqrt{\lambda_k} |e_k\rangle$$

- 4) The matrices $\{A_k\}$ are the smallest set of Kraus operators for \mathcal{E} .

Unitary mixing freedom in Kraus representation

We saw above (§ previously) that multiple sets of Kraus operators can be used to describe the same channel. This can be understood within the Choi representation.

We have $\mathcal{J}(\mathcal{E}) = \sum_i | \text{Vee}(A_i) \rangle\langle \text{Vee}(A_i)|$

But remember this is just a state & we can write it in its eigendecomposition or just as an

ensemble decomposition.

$$\text{i.e. } \mathcal{J}(\mathcal{E}) = \sum_i |\text{Vec}(A_i)\rangle\langle \text{Vec}(A_i)|$$

$$\left. \begin{aligned} &= \sum_k \lambda_k |\lambda_k\rangle\langle \lambda_k| \\ &= \sum_i p_i |\phi_i\rangle\langle \phi_i| \end{aligned} \right\}$$

These are related by $\sqrt{p_i} |\phi_i\rangle = \sum_k u_{ik} \sqrt{\lambda_k} |\lambda_k\rangle$

see lecture 2 *

elements of a unitary/ isometry

* Recap:

$$|\Psi\rangle_{AB} = \sum_u \sqrt{\lambda_u} |\lambda_u\rangle |u\rangle \quad \leftarrow \sum_u \lambda_u |\lambda_u\rangle \langle \lambda_u|$$

Purification of

$$|\Psi'\rangle_{AB} = \sum_i \sqrt{p_i} |\phi_i\rangle |i\rangle \quad \leftarrow \sum_i p_i |\phi_i\rangle \langle \phi_i|$$

Purification of

$$|\Psi'\rangle_{AB} = (I \otimes U) |\Psi\rangle_{AB} \quad \leftarrow \text{As } |\Psi\rangle \text{ & } |\Psi'\rangle \text{ are purifications of the same state.}$$

$$I \otimes \langle i | \Psi\rangle_{AB} = \sqrt{p_i} |\phi_i\rangle = \sum_u \sqrt{\lambda_u} \underbrace{\langle i | U | u \rangle}_{u_{ik}} |u\rangle$$

$$|\text{Vec}(A_i)\rangle = \sqrt{\lambda_i} |\lambda_i\rangle$$

$$|\text{Vec}(B_i)\rangle = \sqrt{p_i} |\phi_i\rangle = \sum_u u_{iu} \sqrt{\lambda_u} |\lambda_u\rangle$$

$$= \sum_u u_{iu} |\text{Vec}(A_u)\rangle$$

$$\sum_{ij} [B_i]_{jj} |\phi_j\rangle = \sum_u u_{iu} [A_u]_{jj} |\phi_j\rangle$$

$$\therefore [B_i]_{jj} = [\sum_k u_{ik} A_k]_{jj}$$

$$\equiv \boxed{B_i = \sum_k u_{ik} A_k}$$

Different possible sets of Kraus operators
are related by unitary/ isometric mixing

Eg. for previous example :

$$B_i = \sum_k u_{ik} A_k$$

$$B_0 = \frac{1}{\sqrt{2}}(A_0 + A_1)$$

$$B_1 = \frac{1}{\sqrt{2}}(A_0 - A_1)$$

$$B_2 = A_2$$

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \quad \begin{matrix} \text{clearly} \\ \text{unitary} \end{matrix}$$

$$B_0 = \frac{1}{\sqrt{3}} \mathbb{1}$$

$$B_1 = \frac{1}{\sqrt{3}} Z$$

$$B_2 = \frac{1}{\sqrt{3}} X$$

$$A_0 = \sqrt{\frac{2}{3}} |10\rangle\langle 01|$$

$$A_1 = \sqrt{\frac{2}{3}} |12\rangle\langle 21|$$

$$A_2 = \frac{1}{\sqrt{3}} X$$